Recap

Quantum Probability:

- **Mixed states**: Density matrix $\rho \succeq 0$, $\mathrm{Tr}(\rho) = 1$
- **POVM**: $E_1 + \cdots + E_k = \mathbb{I}$

$$P[\text{outcome "i"}] = \langle \rho, E_i \rangle$$

- **Observables**: Hermitian X

$$\mathbb{E}_\rho[X] = \langle \rho, X \rangle$$

References:
O'Donnell 18
Quantum Computation & Quantum Information

Lee 21
The art and science of positive definite matrices

von Neumann entropy: $S(\rho) = \mathrm{Tr}\left(\rho \log \frac{1}{\rho}\right)$

Partial trace: $\rho \in \mathbb{C}^{d_A \cdot d_B \times d_A \cdot d_B}$

$$\rho_A = \mathrm{Tr}_B(\rho) \qquad \langle X \otimes \mathbb{I}, \rho \rangle = \langle X, \rho_A \rangle \quad \forall X$$

An example

$$\varphi = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\rho = \varphi\varphi^*$$

$$= \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$S(\rho) = 0$$

$$S(\rho_A) = 1/2 \log 2 + 1/2 \log 2 = 1$$

$$S(\rho_B) = 1$$

What just happened

Classical

$$H(X) \leq H(X,Y)$$

$$I(X;Y) \leq H(X)$$

Quantum

$$S(\rho_A) \overset{?}{\leq} S(\rho) \quad ✗$$

$$\underbrace{S(\rho_A) + S(\rho_B) - S(\rho)}_{\text{Mutual information}} \overset{?}{\leq} S(\rho_A) \quad ✗$$

- Monotonicity of entropy fails due to entanglement

Ex: For any $\rho \in \mathbb{C}^{d\times d}$ $\exists$ $\psi \in \mathbb{C}^{d^2}$ s.t. $\rho' = \mathrm{Tr}_d(|\psi\rangle\langle\varphi|)$
(Purification)

Quantum Relative Entropy

[Umegaki 1962]

$$S(\rho \| \sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$$

$f(x) : I \to \mathbb{R}$ convex

$\Downarrow$

$\mathrm{Tr}(f(M))$ convex for $M = M^*$, $\mathrm{spec}(M) \subseteq I$

- $S(\rho \| \sigma) \geq 0$ with equality iff $\rho = \sigma$
- $S(\rho \| \sigma)$ convex in ρ, σ
- Jointly convex in (ρ, σ) [Araki Lieb 1973] } very nontrivial

$$0 \leq S(\rho \| \rho_A \otimes \rho_B) = \underbrace{\mathrm{Tr}(\rho \log \rho)}_{-S(\rho)} - \underbrace{\mathrm{Tr}(\rho \log (\rho_A \otimes \rho_B))}_{\substack{\mathrm{Tr}(\rho(\log \rho_A \otimes I_B)) \\ + \mathrm{Tr}(\rho(I_A \otimes \log \rho_B))}}$$

$$= -S_\rho + S_{\rho_A} + S_{\rho_B}$$

Mutual info ≥ 0

Data Processing

What does it mean to "process" state ρ ?

Quantum channel :

$$\rho \longrightarrow \underbrace{\rho'}_{\text{density matrix}}$$

linear $\Phi : \mathbb{C}^{d\times d} \to \mathbb{C}^{d'\times d'}$ [Bhatia] : Positive Definite Matrices

- $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$ } Trace Preserving
- $X \succeq 0 \Longrightarrow \Phi(X) \succeq 0$ } Completely Positive map
- Above also hold for $\Phi \otimes I_k$

} CPTP

▸ [Choi, Kraus] : Must have $\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*$. $A_1 \ldots A_n \in \mathbb{C}^{d'\times d}$

$\therefore \sum A_i^* A_i = I$ $\langle I, X\rangle = \mathrm{Tr}(\Phi(X)) = \sum \mathrm{Tr}(A_i X A_i^*) = \langle \sum A_i^* A_i , X\rangle$

Examples

- $\Phi(X) = \mathrm{Tr}(X)$

- $\Phi(X) = \mathrm{Tr}_A(X)$ $\quad X \in M(H_A \otimes H_B)$

- $\Phi(X) = \frac{1}{n}\begin{pmatrix} X & & \\ & \ddots & \\ & & X \end{pmatrix}$, $X \otimes \rho$

- $\Phi(X) = U^* X U$ $\quad$ unitary U

In general $\quad \mathrm{Tr}_B\left(U (X \otimes \rho) U^* \right)$

Quantum Data Processing

- $S(\Phi(\rho) \| \Phi(\sigma)) \leq S(\rho \| \sigma)$ [Consequence of joint convexity]

Consider $\rho_{ABC} \in \mathcal{M}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$

$$S\left(\mathrm{Tr}_B(\rho_{ABC}) \,\|\, \mathrm{Tr}_B(\rho_A \otimes \rho_{BC})\right) \leq S(\rho_{ABC} \| \rho_A \otimes \rho_{BC})$$

$$\cancel{S(\rho_A)} + S(\rho_C) - S(\rho_{AC}) \leq \cancel{S(\rho_A)} + S(\rho_{BC}) - S(\rho_{ABC})$$

$$S(\rho_{ABC}) + S(\rho_C) \leq S(\rho_{AC}) + S(\rho_{BC})$$

Strong subadditivity of von Neumann entropy

- $S(\rho_{ABC}) + S(\rho_C) \leq S(\rho_{AC}) + S(\rho_{BC})$ [Kiefer '59, Robinson Ruelle '66, Lieb Rusaki '73]

$H(X,Y,Z) + H(Z) \leq H(X,Z) + H(Y,Z)$ true classically?

$$0 \leq I(X;Y|Z)$$

What we used in $I(X;Y) \geq I(X;\Phi(Y))$

Holevo's bound

$$\text{Alice} \xrightarrow{\ \sigma_x \in \mathbb{C}^{d \times d}\ } \text{Bob}$$

$X \sim p$ $\qquad\qquad$ $\Phi(\sigma)$

How much information can Bob learn ?

$$\rho = \sum p_x |x\rangle\langle x| \otimes \sigma_x \qquad \rho_A = \sum p_x |x\rangle\langle x| \qquad \rho_B = \sum p_x \sigma_x$$

$$I(\rho_A ; \Phi(\rho_B)) \leq I(\rho_A : \rho_B) = \underbrace{S(\rho_A)}_{H(p)} + \underbrace{S(\rho_B)}_{S(\sum p_x \sigma_x)} - S(\rho)$$

$$\begin{pmatrix} \ddots & & & \\ & \boxed{p_x \cdot \sigma_x} & & \\ & & \boxed{} & \\ & & & \ddots \end{pmatrix}$$

$$S(\rho) = -\sum_x \mathrm{Tr}(p_x \sigma_x \log(p_x \sigma_x))$$

$$= H(p) + \sum_x p_x S(\sigma_x)$$

$$I(\rho_A ; \Phi(\rho_B))$$

$$\leq \underbrace{S\left(\sum p_x \sigma_x\right) - \sum p_x S(\sigma_x)}_{\text{Holevo's } \chi \ \leq \ \log d}$$